Preparation Guide for Test 1

Test 1 will be in class on **Wednesday, 5 February**. This will use the normal class time, so you will have 50 minutes to complete the test. You can get a good idea what to expect on the test by looking at the following practice problems. The test will be shorter than the practice problems—this is meant to give you more problems to practice than you will have on the test, so don't worry if it takes longer than 50 minutes to solve them all. We strongly encourage you to try to problems on your own, before discussing the problems with others. In class on Monday we will discuss any problems students have questions about.

Resources. For the test, you will be permitted to use a single paper page of notes that you prepare and bring to class. It is fine to collaborate with others to prepare your notes. The page should be no larger than a US Letter size page (8.5×11 inches), and you may write (or print) on both sides of the page. You may not use any special devices (e.g., magnifying glasses) to read your page. No other resources, other than your own brain, body, and writing instrument, are permitted during the exam.

Content. The problems on the test will cover material from Classes 1–8, Problem Sets 1 and 2 (including the provided comments), and Chapter 1 of the MCS book. Problems on the test will emphasize things you have seen in at least two of these (Classes, Problem Sets, and MCS Book), and most problems will focus on things that are covered in all three. The purpose of the test is to see how well you understand and can apply the key concepts we have covered so far, not to test your memory of some obscure detail. If you understand the problems on the problem sets and the practice problems in this guide well enough to be able to answer similar questions, you should do well on the exam.

The main topics the test will cover are:

– Propositions, axioms, and proofs (you should understand precisely what each of these are).

– Boolean operators (you should be able to read and interpret a truth table, understand the meaning of IMPLIES ($\implies$), and be able to use truth tables to answer logical questions)

– Inference rules (what an inference rule is, how to determine and show if an inference rule is *sound* or *unsound*, how to correctly use inference rules in a proof).

– Proof methods (you should be able to read and write proofs that use direct proof, contrapositive proof, and proof by contradiction).

For most students, we believe the best way to prepare for the exam will be to (1) go over the problem sets and their solutions, and make sure you understand well any of the problems you did not get before; (2) go through the provided practice problems and try to solve all the problems on your own before getting help and reading the solutions; (3) go through the slides from the lectures and check your understanding of things we have done in class; (4) using your favorite AI assistant to generate more practice problems, get feedback on your solutions, and ask for explanations of proof steps from example proofs; (5) re-reading sections of the book or finding other resources that cover topics you are unsure about, and solving, solving the associated practice problems, especially for any sections on topics where you had difficulty on the problem sets. If you do #1 and #2 and understand well the problems on the practice exam, you should be confident you'll do well on the exam; if you struggled on the problem sets, you would benefit from doing some of #3, #4 (which may be interesting and fun to do for other reasons, but you should be wary to be critical and check understanding of any LLM-produced outputs), and #5 as well.

Practice Problems

Problem 1 *Inference Rules*

For each candidate rule below, indicate whether or not the rule is sound. Support your answer with a convincing argument. The variables P, Q, and R are Boolean propositions (either true or false).

(a) $\dfrac{P, P \implies Q}{Q}$

- ○ Sound
- ○ Not Sound
- ○ Cannot be determined if it is sound or not sound

(b) $\dfrac{P}{Q \implies P}$

- ○ Sound
- ○ Not Sound
- ○ Cannot be determined if it is sound or not sound

(c) $$\frac{P \implies Q, Q \implies P}{P = Q}$$

- ◯ Sound
- ◯ Not Sound
- ◯ Cannot be determined if it is sound or not sound

Problem 2 *Boolean Operators*

Use truth tables to prove De Morgan's law: $\textsc{not}(\textsc{or}(P, Q)) = \textsc{and}(\textsc{not}(P), \textsc{not}(Q))$.

Problem 3 *Bogus Proofs*

Problem 1 on Problem Set 2 asked you to prove the *Not-Even-Odd Conjecture* (stated below) and claimed that there was "an easy way to prove this" (presumably using things we had covered in class). In fact, there is no easy way to prove it!

We regret the error and hope it did not cause you too much trauma! As penance, for this problem we provide several bogus proofs for the conjecture. You should explain for each of the bogus proofs, why it is not a valid proof of the *Not-Even-Odd Conjecture*.

> **Not-Even-Odd Conjecture:** For any natural number n, if n is not even, n is odd.

For these questions you should assume these definitions and theorems:

> **Definition of *even*:** A natural number n is *even* if and only if there exists a natural number k such that $n = 2k$.
>
> **Definition of *odd*:** A natural number n is *odd* if and only if there exists a natural number k such that $n = 2k + 1$.
>
> **Odd-Not-Even Theorem**: For any natural number n, if n is odd, n is not even.

(Note: we are not including answer boxes for these as practice problems, but good answers should fit in a fairly small space.)

(a) Explain why the proof below is not a valid proof of the *Not-Even-Odd Conjecture*. A good answer will identify the first specific proof step that is invalid, and explain why.

1. We are proving for any natural number n, that $P \implies Q$ where $P ::= n$ is not even, and $Q ::= n$ is odd.
2. By the *Odd-Not-Even Theorem*, if n is odd, n is not even.
3. Thus, we can rewrite Q (n is odd) as $Q = n$ is not even.
4. We know for any proposition R, $R \implies R$ is trivially true.
5. So, we can write n is not even $\implies n$ is not even.
6. By (3), we can substitute Q for "n is not even" on the right side of the implication. By (1), we can substitute P for "n is not even" on the left side of the implication. This gives, $P \implies Q$.
7. This is the proposition we are trying to prove, so the proof is complete. □

(b) Explain why the proof below is not a valid proof of the *Not-Even-Odd Conjecture*. A good answer will identify the first specific proof step that is invalid, and explain why.

1. We are proving for any natural number n, that $P \implies Q$ where $P ::= n$ is not even, and $Q ::= n$ is odd.
2. We use proof by contraposition.
3. To show $P \implies Q$ using contraposition, we need to show $NOT(Q) \implies NOT(P)$ and then use the contrapositive inference rule.
4. By the *Odd-Not-Even Theorem*, $NOT(Q) = NOT(n \text{ is odd}) = n$ is even.
5. By the *Odd-Not-Even Theorem*, $NOT(P) = NOT(n \text{ is not even}) = n$ is even.
6. Now we can conclude $NOT(Q) \implies NOT(P)$ since by (4) and (5) they both mean "n is even".
7. Using the contrapositive inference rule, showing $NOT(Q) \implies NOT(P)$ allows us to conclude $P \implies Q$, concluding the proof. □

(c) This time the proof claims that the *Not-Even-Odd Conjecture* is false. A good answer will identify the first specific proof step that is invalid, and explain why. (Note that it would be unfair to ask this question on the test, but the question is really asking you to identify why this would be an unfair question!)

1. We show that the *Not-Even-Odd Conjecture* is false by counter-example.
2. Consider $n = \infty$ (infinity).
3. n is not even since there is no natural number k such that $2k = \infty$.
4. But, n is also not odd, since there is no natural number k such that $2k + 1 = \infty$.
5. Thus, we have a counter-example: n is not even, but n is not odd, so we can conclude that the *Not-Even-Odd Conjecture* is false. □

Problem 4 *Contra-Lemma*

Prove the *Even-Not-Odd Lemma* below:

> **Even-Not-Odd Lemma:** For any natural number n, if n is even, n is not odd.

Problem 5 *Mostly Non-Bogus Propositions (Practice with Irrationals and Proofs)*

For each of the subproblems, *either* (1) prove the given proposition is true or (2) prove the given proposition is false. (Note that each of these subproblems would be a full "problem" on a test. We are just giving you several similar examples for practice.)

(a) *Sum-Evens Proposition*: the sum of any two even numbers is an even number.

(b) *Sum-Irrationals Proposition*: the sum of any two positive irrational numbers must be an irrational number.

(c) *Sum-Ir-Rat Proposition*: the sum of an irrational number and a positive rational number must be an irrational number.

(d) *Product-Ir-Rat Proposition*: the product of an irrational number and a positive rational number must be an irrational number.

(e) *Product-Irrats*: the product of any two irrational numbers must be an irrational number.

