Preparation Guide for Test 3

Test 2 will be in class on **Wednesday, 2 April**. This will use the normal class time, so you will have 50 minutes to complete the test. The test will be similar to Test 1, but focused on new content.

You can get a good idea what to expect on the test by looking at the following practice problems. The test will be shorter than the practice problems—this is meant to give you more problems to practice than you will have on the test, so don't worry if it takes longer than 50 minutes to solve them all. We strongly encourage you to try to problems on your own, before discussing the problems with others. In class on Monday (31 March) we will discuss any problems students have questions about.

Resources. For the test, you will be permitted to use a single paper page of notes that you prepare and bring to class. It is fine to collaborate with others to prepare your notes. The page should be no larger than a US Letter size page (8.5×11 inches), and you may write (or print) on both sides of the page. You may not use any special devices (e.g., magnifying glasses) to read your page. No other resources, other than your own brain, body, and writing instrument, are permitted during the exam.

Content. The problems on the test will cover material from Classes 1–29 (but focus more on material covered since Test 2), Problem Sets 1–6 including the provided comments (with a focus on Problem Set 5 and 6), and the MCS readings mentioned in the problem set Preparation sections. Problems on the test will emphasize things you have seen in at least two of these (Classes, Problem Sets, and MCS Book), and most problems will focus on things that are covered in all three. The purpose of the test is to see how well you understand and can apply the key concepts we have covered so far, not to test your memory of some obscure detail. If you understand the problems on the problem sets and the practice problems in this guide well enough to be able to answer similar questions, you should do well on the exam.

The main topics the test will cover are:

- Ordered and Well-Ordered — determining if sets are ordered or well-ordered under a given comparator
- Well-Ordering Proofs — understand how to write and spot bugs in Well-Ordering Proofs
- Induction Principle — understanding the principle of induction, both in general and the specializations we have seen, and how to use it in proofs (as well as how to identify common pitfalls in incorrectly using induction in bogus proofs).

For most students, we believe the best way to prepare for the exam are the same as those we described on the Preparation Guide for Test 1.

Practice Problems

Problem 1 *Well and Unwell Ordered Sets*

Recall these definitions from Class 20:

> **Definition of ordered:** A set S is *ordered* with respect to some comparator, $<: S \times S \to Boolean$, and equality operator, $=: S \times S \to Boolean$, iff $\forall a, b, c \in S$:
>
> 1. $\neg(a = b) \implies (a < b) \vee (b < a)$.
> 2. $(a < b) \wedge (b < c) \implies a < c$.
>
> **Definition of well ordered:** An ordered set S with comparator $<$ and equality operator $=$, is *well ordered* iff all of its non-empty subsets have a minimum element:
>
> $$\forall T \subseteq S \,.\, T \neq \emptyset \implies \exists m \in T \,.\, \forall t \in T \,.\, t \neq m \implies m < t.$$

For each subproblem below, answer if the given set is **well ordered** with respect to the given comparator. Note that in order to be well ordered, the set must be ordered. You may assume the standard equality operator, $=$, is used for all of the orderings.

Support your answers with a brief, but clear and convincing, argument.

(a) $\{n \in \mathbb{N} \mid n \text{ is divisible by } 5\}$; $<$.

(b) $\{z \in \mathbb{Z} \mid z \leq 0\}$; $<$.

(c) $\{\ \{k \mid k \in \mathbb{N}, k \leq n\}\ \mid n \in \mathbb{N}\ \}$; $smaller(a, b) ::= |\,a\,| < |\,b\,|$.

(d) $\{s \mid s \in$ students in DMT1$\}$; $younger(a, b) ::= a$ was born before b.

(e) $\{s \mid s \in$ students in DMT1$\}$; $solverer(a, b) ::= a$ has earned a higher score than b on a test in cs2120.

(f) $\mathbb{N}$; $fairer(n) ::=$ *takeaway*(n, true) is a better game for Player 1 than for Player 2.

Problem 2 *Factorial Sum (MCS 5.14)*

(a) Use the **well-ordering principle** to prove:

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1. \tag{5.16}$$

(b) Use the **principle of induction** to prove the same theorem.

Problem 3 *Cartesian Product Cardinality*

Suppose A and B are finite sets. Prove by induction that $|A \times B| = |A| \cdot |B|$ where $A \times B$ is the cartesian product of A and B. (Hint: you can apply the induction over the size of A, but make sure to clearly define the induction hypothesis.)

Problem 4 *Minimum Rational*

Prove that every non-empty finite set of rational numbers has a minimum element. (You should be able to prove this using either the well ordering principle or the principle of induction, and also be able to explain why this does not contradict the fact that the rational numbers are not well ordered.)

Problem 5 *Exponential Losses*

The "Exponential Losses" Casino has chips with value \$1, \$2, \$4, \$8, \$16, $\ldots$, $\$2^k$, but has a rule that bettors may not use more than one of the same value of chip to make any bet.

(a) Use the **well-ordering principle** to prove that all integer bets from $\$1$ to $\$2^{k-1}$ can be made.

(b) Use the **principle of induction** to prove that all integer bets from $\$1$ to $\$2^{k-1}$ can be made.

Problem 6 *Fair Takeaway*

In class, we proved a theorem about when Player 1 has a winning strategy in takeaway. Prove that if Player 1 does not have a winning strategy, Player 2 does.

Problem 7 *Power Takeaway*

$(\star)$[1] Consider this variation on the *takeaway* game from class: instead of removing $k \in \{1, 2, 3\}$ sticks each turn, a player must remove $k \in \{2^z \mid z \in \mathbb{N} \wedge 2^z \leq n\}$ (where n is the number of sticks remaining) sticks.

Devise and prove a theorem similar to the one we derived in class for the 1-2-3 takeaway game that states when Player 1 has a winning strategy.

[1]The star means we think this may be an unreasonably hard problem and you shouldn't be worried if you can't answer it.

