Preparation Guide for Test 4

Test 4 will be in class on **Wednesday, 23 April**. This will use the normal class time, so you will have 50 minutes to complete the test. The test will be similar to the previous tests, with the same rules about resources. You can get a good idea what to expect on the test by looking at the following practice problems. The test will be shorter than the practice problems—this is meant to give you more problems to practice than you will have on the test, so don't worry if it takes longer than 50 minutes to solve them all. We strongly encourage you to try to problems on your own, before discussing the problems with others.

Resources. (Same as Tests 1–3, just restated here) For the test, you will be permitted to use a single paper page of notes that you prepare and bring to class. It is fine to collaborate with others to prepare your notes. The page should be no larger than a US Letter size page (8.5×11 inches), and you may write (or print) on both sides of the page. You may not use any special devices (e.g., magnifying glasses) to read your page. No other resources, other than your own brain, body, and writing instrument, are permitted during the exam.

Content. The problems on the test will cover material from Classes 1–39, Problem Sets 1–8 including the provided comments (with a focus on Problem Set 6, 7 and 8), Tests 1–3 and the comments provided for those tests, and the MCS readings mentioned in the problem set Preparation sections. It will focus on material covered since Test 3, but you should expect it to include some problems to assess your understanding of the principle of induction and ability to use it in proofs, as well as to test comprehensive understanding and ability to apply concepts we have covered through the semester. more on material covered since Test 3 and Induction), Problems on the test will emphasize things you have seen in at least two of these (Classes, Problem Sets, previous Tests and Preparation Guides, and the MCS Book), and most problems will focus on things that are covered in at least three of these. The purpose of the test is to see how well you understand and can apply the key concepts we have covered in the course, not to test your memory of some obscure detail. If you understand the problems on the problem sets and the practice problems in this guide well enough to be able to answer similar questions, you should do well on the exam.

The main topics the test will emphasize are:

– Induction — understanding the principle of induction, both in general and the specializations we have seen, and how to use it in proofs (as well as how to identify common pitfalls in incorrectly using induction in bogus proofs). Being able to write induction proofs involving infinite sets.

– Infinite Sets — understanding the definitions of finite and infinite sets, determining if a set is infinite or finite, being able to prove a set is finite or infinite, understanding the difference between potential and actual infinities.

– Countability and uncountability — understanding the definitions of countable and uncountable, being able to determine if a set is countable or uncountable (and proving that), using bijections, surjective functions, and diagonalization in proofs of countability and uncountability.

For most students, we believe the best way to prepare for the exam are the same as those we described on the Preparation Guide for Test 1.

Practice Problems

Problem 1 *Countability*

For each subproblem, answer if the set S described is *finite*, *countably infinite* or *uncountable* and give a brief explanation why

(a) $S := \mathbb{R} - \mathbb{N}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(b) $S := \{s \mid s \text{ is a student in this class } \wedge \ s \text{ has a name that rhymes with "Cantor"}\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(c) $S := \{\sum_{i=0}^{k} i \mid k \in \mathbb{N}\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(d) $S := \{G \in pow(\mathbb{N} \times \mathbb{N}) \mid R = (\mathbb{N}, \mathbb{N}, G)$ is a bijection$\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(e) $S := \{R \mid R = (\mathbb{N}, \{0,1\}, G \subseteq \mathbb{N} \times \{0,1\})$ is a function$\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(f) $S := \{r \mid r \in \mathbb{R} \wedge r \geq 20 \ \wedge \ r < 20.5\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(g) $S := \{r \mid r \in \mathbb{R} \wedge \sin(r) = 0\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(h) $S := \{r \mid r \in \mathbb{R} \wedge \sin(r) > \cos(r)\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

Problem 2 *Relations*

For each subproblem, answer is the given relation would be a valid relation to use in a proove that $\mathbb{Z}$ is *countably infinite* (without anything else in the proof other than an argument that explains the property the given relation has).

(a) $R = (\mathbb{N}, \mathbb{Z}, G = \{(a, b) \mid a \in \mathbb{N}, b = a\})$

Justify your answer with a brief but clear and convincing explanation:

(b) $R = (\mathbb{N}, \mathbb{Z}, G = \{(a, b) \mid a \in \mathbb{N};\ \text{if } \exists k.a = 2k,\ b = k;\ \text{otherwise } a \text{ is odd so } \exists k\,.\,a = 2k + 1, b = -k)\})$

Justify your answer with a brief but clear and convincing explanation:

(c) $R = (\mathbb{Z}, \mathbb{N}, G = \{(a, b) \mid \forall a \in \mathbb{Z},\ \text{if } a \geq 0, b = a; \text{otherwise } b = -a\})$

Justify your answer with a brief but clear and convincing explanation:

(d) $(\star)$ $R = (\mathbb{Z}, \mathbb{Q}, G = \{(a, b) \mid \forall a \in \mathbb{Z}, b = \frac{a}{a+1}\})$ (assume we do already know that $\mathbb{Q}$ is countably infinite)

Justify your answer with a brief but clear and convincing explanation (hint: consider -1 carefully!):

Problem 3 *Diagonalization Proof?*

In class we saw a diagonalization proof (very close to the one Cantor developed in his 1891 paper) that the set of infinite binary strings, $\{0,1\}^\infty$ is *uncountable*:

1. We prove the set of infinite binary strings, $S = \{0,1\}^\infty$ is uncountable by contradiction.
2. Assume towards a contradiction that there exists a surjective function $R = (\mathbb{N}, S, G \subseteq \mathbb{N} \times S)$ from $\mathbb{N}$ to S.
3. Since R is function, for each $n \in \mathbb{N}$ there is at most one $s \in S$ where $(n, s) \in G$. Let $A_n := s$ be that element and identify the characters of A_n as $a_{n,1}a_{n,2}a_{n,3}\ldots$.
4. Define $b = b_1b_2b_3\ldots$ where $b_i = \neg a_{(i,i)}$ and $\neg 0 = 1$ and $\neg 1 = 0$.
5. Since b is an infinite bitstring, we know $b \in S$. Since each bit is different from one bit in each A_n and since R is surjective we know $\forall s \in S.\exists m \in \mathbb{N}.A_m = s$ but A_m is not in the mapping and we have a contradiction.

For each subproblem, indicate of the alternate construction for step 4 would result in a valid or invalid proof and explain why. (Differences from the previous subproblem are highlighted in red.)

(a) Define $b = b_1b_2b_3\ldots$ where $b_i = W(a_{i,i})$ where $W(0) = 1$ and $W(1) = 1$.

Justify your answer with a brief but clear and convincing explanation:

(b) Define $b = b_1b_2b_3\ldots$ where $b_i = W(a_{i,i})$ where $W(0) = 1$ and $W(1) = 1$.

Justify your answer with a brief but clear and convincing explanation:

(c) Define $b = b_1b_2b_3\ldots$ where $b_i = W(a_{i,i})$ where $W(0) = 1$ and $W(1) = 2$.

Justify your answer with a brief but clear and convincing explanation:

(d) Define $b = b_1 b_2 b_3 \ldots$ where $b_i = W(a_{1,i})$ where $W(0) = 1$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

(e) Define $b = b_1 b_2 b_3 \ldots$ where $b_i = W(a_{2i,i})$ where $W(0) = 1$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

(f) Define $b = b_1 b_2 b_3 \ldots$ where $b_i = W(a_{i,2i})$ where $W(0) = 1$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

Problem 4 *Truths and Falsehoods*

For each subproblem, decide if the proposition stated is true or false. If it is true, provide a convincing proof. If it is false, either provide a counter-example or explain convincingly why it is false some other way.

(a) For all sets A and B where $B \subseteq A$, if B is uncountable, then A is *uncountable*.

◯ True

◯ False

◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

(b) For all countable sets A, $\mathbb{R} - A$ is *uncountable*.

◯ True

◯ False

Justify your answer with a brief but clear and convincing explanation:

(c) For all uncountable sets A, $\mathbb{R} - A$ is *uncountable*.

◯ True

◯ False

Justify your answer with a brief but clear and convincing explanation:

(d) For all uncountable sets A, $pow(\mathbb{R}) - A$ is *uncountable*.

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

(e) For all uncountable sets A, $|pow(A)| > |\mathbb{R}|$.

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

(f) ($\star\star?$[1]) If there exists a set A where $|A| > |pow(\mathbb{R})|$ and $|A| < |pow(pow(\mathbb{R}))|$ then Laufey's best song is "Beautiful Stranger".

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

[1] This means don't worry about this one for preparing for the exam!

Problem 5 *Cardinality Proofs*

(a) Prove that $\forall n \in \mathbb{N}. |\mathbb{N}_n \times \{0, 1\}| = 2n$.

(b) Prove that $\forall n \in \mathbb{N}. \{k \mid k \in \mathbb{N}, k > n\}$ is a *countably infinite* set.

(c) Prove that $\forall n \in \mathbb{N}$ the number of bijections between $\mathbb{N}_n$ and $\mathbb{N}_n$ is $n!$ (recall that $n! = \Pi_{i=1}^{n} i$).

(d) Prove that the number of bijections between $\mathbb{N}$ and $\mathbb{N}$ is uncountable.

