Problem Set 3: Comprehending Sets

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Submit your responses as a single PDF file to gradescope before **8:29pm** on **Thursday, 13 February**.

Collaboration and Resources Policy (identical to PS2 but repeated here)
Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from last year's offering of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions.

Preparation

This problem set includes material covered in Section 4.1 (Sets) and 3.6 (Predicate Formulas) of the *MCS book*, and focuses on the new concepts introduced in Classes 10–11 (but also builds upon everything we have done so far this semester).

Directions

(Identical to PS2, other than the template URL)

1. Follow the steps as in Problem Set 2 to create your own copy of the template in https://www.overleaf.com/read/tcvjnjjpmjrr#db92dc.
2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.
3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.
4. Replace the second line in ps3.tex, `\usepackage{dmt}` with `\usepackage[response]{dmt}` so the directions do not appear in your final PDF.
5. Download your complete ps3.pdf file, and submit it using gradescope.

Problem 1 *Notable Sets*

We use the notation $\mathbb{N}$ to mean the set of all the natural numbers ($\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, which is not precisely defined this way since we need to hope you can guess what the $\ldots$ means) and $\mathbb{Q}$ to mean the set of all the rational numbers (you should already know the definition of a *rational* number).

(a) Which of the sets below are equal to $\mathbb{N}$? Check *all* of that choices that are equal to $\mathbb{N}$.

- ☐ $\{n \mid n \in \mathbb{N}\}$
- ☐ $\mathbb{N} \cap \mathbb{Q}$
- ☐ $(\mathbb{N} \cup \mathbb{Q}) - \mathbb{Q}$
- ☐ $\{n \in \mathbb{N} \mid n \text{ is even } \} \cup \{n \in \mathbb{N} \mid n \text{ is odd } \} \cup \{n \in \mathbb{N} \mid n \text{ is prime } \}$

(b) Which of the sets below are empty? Check *all* of that choices that are equal to the empty set, $\{\ \}$.

- ☐ $\{n \in \mathbb{N} \mid n \notin \mathbb{Q}\}$
- ☐ $\{n \in \mathbb{N} \mid n^2 < n\}$
- ☐ $\{n \in \mathbb{N} \mid n \text{ is even } \} \cap \{n \in \mathbb{N} \mid n \text{ is prime } \} \cap \{n \in \mathbb{N} \mid n > 2\}$
- ☐ $\{\{r\} \cup \{r, r+1\} \mid r \in \mathbb{Q} \wedge r^2 = 2\}$

(c) Which one of the sets below is equal to $\mathbb{N}$? (Check the one correct answer and explain why in the box below.) The notation $|\,S\,|$ means intuitively the number of elements in set S. For example, $|\{0, 1, 2\}| = 3$. (We will defined set cardinality more carefully soon, but this intuitive notion should be all you need for this problem.)

- ◯ $\{\,|\{k \mid k \in \mathbb{N}, k < n\}|\ \mid n \in \mathbb{N}\,\}$
- ◯ $\{\,|\{k \mid k \in \mathbb{N}, k \leq n\}|\ \mid n \in \mathbb{N}\,\}$

Explain why to convince us you didn't just make a lucky guess:

Problem 2 *Defining Set Operations*

The set A is a *proper subset* of the set B (commonly denoted as $A \subsetneq B$ or $A \subset B$) if and only every element of A is also an element of B and A is smaller than B. Without using any notion of set size (that is, not assuming an operator that can compare the sizes of two sets or that gives the size of an input set), provide a precise definition of $\subsetneq$. Your definition should be similar in style and formality to the set operation definitions we saw in Class 10.

Problem 3 *Quantifiers*

(a) State if the proposition below is true or false, and support your answer:

$$\forall x \in \mathbb{N} \,.\, \exists y \in \mathbb{N} \,.\, \exists b \in \{0,1\} \,.\, x = 2y + b.$$

(b) What is the negation of the proposition below, where Q and R are Boolean predicate:

$$P ::= \forall x \in \mathbb{N} \,.\, Q(x) \implies R(x).$$

Problem 4 *Proving Set Equalities*

(Problem 4.8 in the MCS book) For any sets A, B, and C, prove:

$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$$

End of Problem Set 3!

Remember to follow the instructions to prepare and submit your PDF (including using *[response]* to remove the directions and completing *\collaborators* with information on your collaborators and the resources you used.