Problem Set 4: Finite Set Cardinality

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Submit your responses as a single PDF file to Gradescope before **8:29pm** on **Thursday, 20 February**.

Preparation

This problem set includes material covered in Chapter 4 of the *MCS book*, and focuses on the new concepts introduced in Classes 12–14 (but also builds upon everything we have done so far this semester).

Collaboration and Resources Policy (identical to PS2 but repeated here)
Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from last year's offering of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions.

Directions

(Identical to PS2 and PS3, other than the template URL)

1. Follow the steps as in Problem Set 2 to create your own copy of the template in https://www.overleaf.com/read/kdfyffygvwpn#1f06d7.
2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.
3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.
4. Replace the second line in ps4.tex, `\usepackage{dmt}` with `\usepackage[response]{dmt}` so the directions do not appear in your final PDF.
5. Download your complete ps4.pdf file, and submit it using gradescope.

Problem 1 *Simple Sets*

Write the value of each of the following expressions. The notation *pow* denotes the *powerset* as defined below:

> For any sets A and B, $B \in \textit{pow}(A)$ iff $B \subseteq A$.

(a) $\{1, 3\} \times \{0, 1, 2\} =$

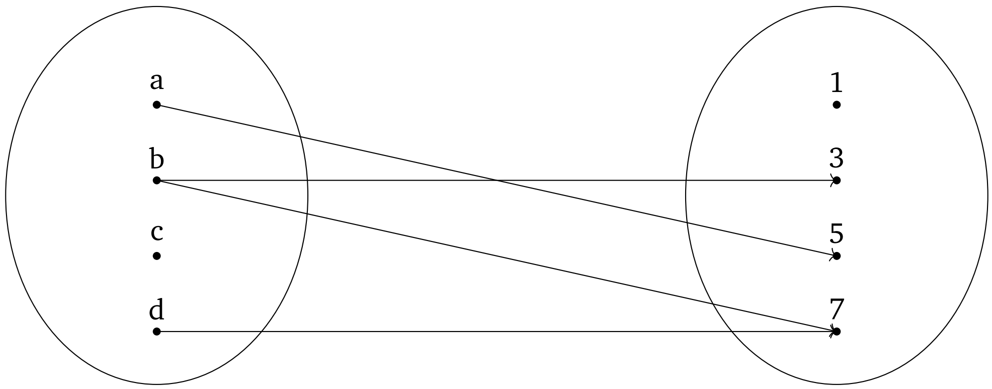
(b) $\{0, 1, 2\} \times \{\} =$

(c) $\textit{pow}(\{\}) =$

(d) $\textit{pow}(\{0, 1, 2\}) =$

Problem 2 *Binary Relations*

For each subproblem, answer with a single edge which should be added or removed from the graph to make the binary relation have the stated property. You may assume $\{a, b, c, d\}$ is the entire domain and $\{1, 3, 5, 7\}$ is the entire codomain, and the graph is depicted by the diagram below. Each subpart is independent (e.g., a change you make to the graph for part (b) does not impact the graph for part (c)).



(a) **Example:** total

Add $(c, 3)$.

All you need to write would be the text above, but here is an explanation: We can make the binary relation **total** by adding the edge $(c, 3)$ to the graph because then all of the points in the domain have ≥ 1 arrows out.

(b) function

(c) surjective

(d) Explain why is it not possible to make the relation bijective by adding or removing a single edge from the graph.

Problem 3 *Patching a Proof (MCS Problem 4.15)*

(a) Give a counterexample that shows the proposition below is false, with a convincing explanation of why your counterexample invalidates the proposition.

Bogus Proposition. For any finite sets A, B, C, D, let

$$L ::= (A \cup B) \times (C \cup D),$$

$$R ::= (A \times C) \cup (B \times D).$$

Then $L = R$.

(b) Identify the mistake in the following proof of the Bogus Proposition.

Bogus proof. Since L and R are both sets of pairs, it's sufficient to prove that

$$(x, y) \in L \iff (x, y) \in R \quad \text{for all } x, y.$$

The proof is a direct proof using a chain of iff implications:

$$\begin{aligned}
(x, y) \in R &\iff (x, y) \in (A \times C) \cup (B \times D) \\
&\iff (x, y) \in A \times C \text{ or } (x, y) \in B \times D \\
&\iff (x \in A \text{ and } y \in C) \text{ or } (x \in B \text{ and } y \in D) \\
&\iff \text{either } x \in A \text{ or } x \in B, \text{ and either } y \in C \text{ or } y \in D \\
&\iff x \in A \cup B \text{ and } y \in C \cup D \\
&\iff (x, y) \in L. \quad \square
\end{aligned}$$

(c) Explain how to modify the bogus proof of the bogus proposition to produce a correct proof of a modified proposition where the conclusion is changed to $R \subseteq L$.

Problem 4 *Injective Proof*

Prove that the binary relation $R = (\mathbb{R}, \mathbb{R}, G = \{(x, \sin(x))|x \in \mathbb{R}\})$ is not *injective*. The notation $\mathbb{R}$ denotes the set of real numbers. Recall that a binary relation is *injective* if and only if each element in the codomain has ≤ 1 arrows in.

Problem 5 *Sizing Sets*

Using the definition of cardinality from class (repeated below), prove that for any two finite sets, A and B, if $|A \cup B| \neq |A|$ then $B - A$ is non-empty. We use $|S|$ to denote the cardinality of set S.

> **Definition of Cardinality:** The *cardinality* of the set $\mathbb{N}_k = \{n \in \mathbb{N} \mid n < k\}$ is k. Two sets have the same cardinality if and only if there is a bijection between two sets.

End of Problem Set 4!

Remember to follow the instructions to prepare and submit your PDF (including using *[response]* to remove the directions and completing *collaborators* with information on your collaborators and the resources you used.

