Problem Set 5: Well Ordering

Response by: **TODO: replace this with your name (and computing id)**

Submit your responses as a single PDF file to gradescope before **8:29pm** on **Thursday, 6 March**.

Preparation

This problem set includes material covered in Chapter 2 (*The Well Ordering Principle*) of the *MCS book*, and focuses on the new concepts introduced in Classes 18–21 (but also builds upon everything we have done so far this semester).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Collaboration and Resources Policy

Note: identical to previous problem sets but repeated here.

Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from previous offerings of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions.

Directions

(Almost identical to PS4, except there is no need to change the `[responses]` in the template.)

1. Follow the steps as in previous problem sets to create your own copy of the template in https://www.overleaf.com/read/mnpgvnvtyxbq#03c118.
2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.
3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.
4. Download your complete ps5.pdf file, and submit it using gradescope.

Problem 1 *Well and Unwell Ordered Sets*

Recall these definitions from Class 20:

> **Definition of ordered:** A set S is *ordered* with respect to some comparator, $<: S \times S \to Boolean$, and equality operator, $=: S \times S \to Boolean$, iff $\forall a, b, c \in S$:
>
> 1. $\neg(a = b) \implies (a < b) \vee (b < a)$.
> 2. $(a < b) \wedge (b < c) \implies a < c$.
>
> **Definition of well ordered:** An ordered set S with comparator $<$ and equality operator $=$, is *well ordered* iff all of its non-empty subsets have a minimum element:
>
> $$\forall T \subseteq S \,.\, \exists m \in T \,.\, \forall t \in T \,.\, t \neq m \implies m < t.$$

For each subproblem below, answer if the given set is **well ordered** with respect to the given comparator. Note that in order to be well ordered, the set must be ordered. You may assume the standard equality operator, $=$, is used for all of the orderings.

Support your answers with a brief, but clear and convincing, argument.

(a) $EVENS = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\}$; $<$.

(b) $\emptyset$; $<$.

(c) $\mathbb{N}$; $>$.

(d) $\mathit{pow}(\mathbb{N})$; $smaller(a, b) ::= |\, a \,| < |\, b \,|$.

(e) $\{ \left| \{k \mid k \in \mathbb{N}, k < n\} \right| \mid n \in \mathbb{N} \}$; $smaller(a, b) ::= |\, a \,| < |\, b \,|$.

Problem 2 *Well Ordering Proofs*

The template for Well-Ordering Proofs (introduced in MCS, Section 2.2, slightly restated here) is:

To prove that $P(n)$ is true for all $n \in \mathbb{N}$:

1. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} \mid \neg(P(n))\}$.
2. Assume towards a contradiction that C is non-empty.
3. By the well-ordering principle, there must be a minimum element, $m \in C$.
4. Reach a contradiction (this is the creative part!). One way to reach a contradiction would be to show $P(m)$. Another way is to show there must be an element $m' \in C$ where $m' < m$.
5. Since C must be empty, there are no counter-examples and $P(n)$ always holds.

There are two main options for step 4: proving $P(m)$, which contradicts $m \in C$, or proving $\neg P(m) \implies \exists m' < m . \neg P(m')$ which contradicts that m is the minimum element of C since m' would be a smaller element in C.

Consider the proof in the MCS book for Theorem 2.3.1: *Every positive integer greater than one can be factored as a product of primes.* which we have reproduced with numbering and slight rewording here:

1. Let C be the set of all integers greater than one that cannot be factored as a product of primes.
2. We assume C is not empty and derive a contradiction.
3. If C is not empty, there is a least element $m \in C$ by the Well Ordering Principle.
4. This m cannot be prime, because a prime by itself is considered a (length one) product of primes, and no such products are in C.
5. So m must be a product of two integers a and b where $1 < a < m$ and $1 < b < m$.
6. Since a and b are smaller than the smallest element in C, we know that $a \notin C$ and $b \notin C$. Thus, a can be written as a product of primes $p_1 \cdot p_2 \cdots p_k$ and b as a product of primes $q_1 \cdots q_l$.
7. Therefore, $m = p_1 \cdot p_2 \cdots p_k \cdot q_1 \cdots q_l$, which shows can be written as a product of primes.
8. This contradicts the claim that $m \in C$. Our assumption that C is not empty must therefore be false, so we know the proposition that every positive integer greater than one can be factored as a product of primes holds for all positive integers.

(a) Which of these options does this proof use:

◯ It reaches a contradiction by showing $P(m)$.

◯ It reaches a contradiction by showing there is an element $m' \in C$ where $m' < m$.

◯ Neither of these, it is not following the template above.

(b) How do we know the claim in Step 5 is correct? (There are two parts to this claim: (1) that m is the product of two integers, (2) that both of the integers are in the range $(1, m)$.)

Problem 3 *Even Fibonacci Numbers*

Problem 2.2 in the MCS book. Your answer should clearly identify the incorrect step, and explain why it is incorrect.

Problem 4 *Redemption for Problem Set 2*

We asked you to prove this theorem on Problem Set 2, but it was not a reasonable problem given what we had covered up to that point. Now you should be able to prove it using the well-ordering principle.

> **Not-Even-Odd Conjecture:** For any natural number n, if n is not even, n is odd.

You may find some of the following definitions useful:

> **Definition of Odd:** An integer, z, is odd if and only if there exists an integer k such that $z = 2k + 1$.
>
> **Definition of Even:** An integer, z, is even if and only if there exists an integer k such that $z = 2k$.
>
> **Even-Not-Odd Lemma:** For any natural number n, if n is even, then n is not odd.
>
> **Odd-Not-Even Lemma:** For any natural number n, if n is odd, then n is not even.

You may (for full expected credit) use the familiar notion of natural numbers and definitions of even and odd above, but we'll be more impressed if you use the set theoretical definition of natural numbers (from Class 17) and provide a definitions of even and odd that work for that definition (where we do not have a definition of multiplication yet, so they cannot use expressions like $2k$).

For this problem you do not need to put your answer in an answer box, just write your answer below (and feel free to go onto the next page if you want).

Proof of the Not-Even-Odd Conjecture:

End of Problem Set 5!

Remember to follow the instructions to prepare and submit your PDF (which unlike the previous problem sets should still include all of the directions, and start with your answers on page 3) and remember to complete *collaborators* with information on your collaborators and the resources you used.