Problem Set 6: Induction

Response by: **TODO: replace this with your name (and computing id)**

Submit your responses as a single PDF file to gradescope before **8:29pm** on **Thursday, 27 March**.

Preparation

This problem set includes material covered in Chapter 5 (*Induction*) of the *MCS book*, and focuses on the new concepts introduced in Classes 24–26 (but also builds upon everything we have done so far this semester).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Collaboration and Resources Policy

Note: identical to previous problem sets but repeated here.

Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from previous offerings of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions.

Directions

(Almost identical to PS5)

1. Follow the steps as in previous problem sets to create your own copy of the template in https://www.overleaf.com/read/yjrqbnkbypmv#6bb8bc.
2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.
3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.
4. Download your complete ps6.pdf file, and submit it using gradescope.

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Problem 1 *False Theorem (MCS 5.7)*

False Theorem. For all $n \geq 0$,

$$2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

1. We use induction.
2. Let $P(n)$ be the proposition that
$$2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$
3. **Base case:** $P(0)$ is true, since both sides of the equation are equal to zero. (Recall that a sum with no terms is zero.)
4. **Inductive step:** Now we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. So suppose that $P(n)$ is true; that is,
$$2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$
5. Then we can reason as follows:
$$\begin{aligned} 2 + 3 + 4 + \cdots + n + (n+1) &= [2 + 3 + 4 + \cdots + n] + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$
6. Above, we group some terms, use the assumption $P(n)$, and then simplify. This shows that $P(n)$ implies $P(n+1)$.
7. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Where exactly is the error in this proof?

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Problem 2 *Sums of Cubes (based on MCS 5.3)*

(a) Rewrite the following expression in summation notation

$$0^3 + 1^3 + 2^3 + \cdots + n^3$$

(b) Use induction to prove that for all natural numbers, $n \in \mathbb{N}$,

$$0^3 + 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

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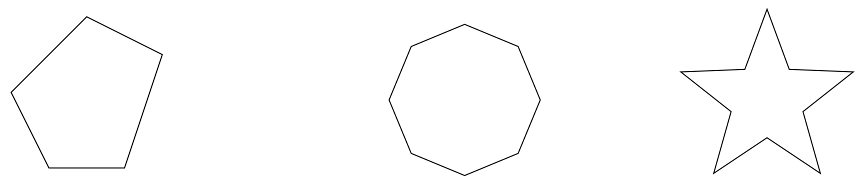
Problem 3 *Convex Polygons*

A convex polygon is a polygon where all line segments connecting any two points in the polygon are fully contained in the polygon. For example, of the three polygons below, the left two are convex, but the rightmost one is not.



convex **convex** **non-convex**

Use the principle of induction to prove that any convex polygon with n sides can be divided into $n-2$ triangles.

Proof:

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Problem 4 *Redemption Redux*

We asked you to prove this theorem on Problem Set 2, but it was not a reasonable problem given what we had covered up to that point, and on Problem Set 5, using the Well-Ordering Principle. Now, we want you to prove the same theorem using the Principle of Induction.

> **Not-Even-Odd Conjecture:** For any natural number n, if n is not even, n is odd.

You should (for full expected credit) use the familiar notion of natural numbers and definitions of even and odd above (and we don't encourage you to use the set theoretic notion of the natural numbers).

You may find some of the following definitions useful:

Definition of Odd: An integer, z, is odd if and only if there exists an integer k such that $z = 2k + 1$.

Definition of Even: An integer, z, is even if and only if there exists an integer k such that $z = 2k$.

Even-Not-Odd Lemma: For any natural number n, if n is even, then n is not odd.

Odd-Not-Even Lemma: For any natural number n, if n is odd, then n is not even.

For this problem you do not need to put your answer in an answer box, just write your answer below (it is fine if it goes onto the next page). You cannot use the Well-Ordering Principle for this proof, and should use the Principle of Induction on the Natural Numbers.

Proof of the Even-Odd Conjecture using the Principle of Induction:

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End of Problem Set 6!

Remember to follow the instructions to prepare and submit your PDF (which like PS5 should still include all of the directions and start with your answers on page 3) and remember to complete *collaborators* with information on your collaborators and the resources you used.

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