Problem Set 8: (Un)countable Sets

Response by: **TODO: replace this with your name (and computing id)**

Submit your responses as a single PDF file to gradescope before **8:29pm** on **Thursday, 17 April**.

Preparation

This problem set includes material covered in Chapter 8 (including the parts on uncountable sets that were not included in Problem Set 7) of the *MCS book*, and focuses on the concepts discussed in Classes 32–36 (but also builds upon everything we have done so far this semester).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Collaboration and Resources Policy

Note: identical to previous problem sets but repeated here.

Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from previous offerings of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions.

Directions

(Almost identical to PS5)

1. Follow the steps as in previous problem sets to create your own copy of the template in Thttps://www.overleaf.com/read/kvgvzvcddmbg#0a0ff9.
2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.
3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.
4. Download your complete ps8.pdf file, and submit it using gradescope.

Problem 1 *Countability*

For each subproblem, answer if the set S described is *countable* or *uncountable* and give a brief explanation why

(a) $S := \mathbb{R} - \mathbb{Z}$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(b) $S := \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots\}$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(c) $S := \mathit{pow}(\{1, 2, 3\})$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(d) $S := pow(\mathbb{Z})$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(e) $S := \{p \mid p \text{ is a valid Python program}\}$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(f) $S := \{z \mid z \text{ is an irrational number}\}$

- ◯ S is countable
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

Problem 2 *Countable Monsters*

Prove that the set $S := \mathbb{N} \cup \{Dracula, Cantorula, Godzilla\}$ is *countably infinite*. You should give a precise and formal definition of any binary relation you use in your proof.

Proof:

Problem 3 *Pairs of Integers*

Prove that the set $\mathbb{Z} \times \mathbb{Z}$ is *countably infinite*.

Proof:

Problem 4 *Really Uncountable*

Show that $\mathbb{R} - \{\frac{x}{7} \mid x \in \mathbb{N}\}$ is *uncountable*.

Proof:

Problem 5 *Finite Permutations*

A *permutation* is a linear ordering of the elements in a set. For example, for the set $\{1, 2, 3\}$ there are 6 ($3 \times 2 \times 1 = 6$) permutations: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

Prove that for all $k \in \mathbb{N}$, the set $\mathbb{N}_k$ has $k!$ permutations (where $k! = \Pi_{i=1}^{k} i$).

Proof:

Problem 6 *Infinite Permutations*

($\star$) Determine if the cardinality of the set of all permutations of $\mathbb{N}$ is *countable* or *uncountable*, and support your answer with a convincing proof.

Theorem: The set of all permutations of $\mathbb{N}$ is *which?*.

Proof:

End of Problem Set 8: (Un)countable Sets!

Remember to follow the instructions to prepare and submit your PDF (which like PS6 should still include all of the directions and start with your answers on page 3) and remember to complete *collaborators* with information on your collaborators and the resources you used (including describing and linking to a transcript of any AI tool use for each problem where you used an AI tool).