Test 1 — Comments

Problem 1 *Inference Rules*

For each candidate rule below, indicate whether or not the rule is sound. Support your answer with a convincing argument. The variables P and Q are Boolean propositions (either true or false).

(a) $$\frac{\text{NOT}(\text{NOT}(\text{NOT}(P)))}{P}$$

Comments: Not Sound. The antecedents simplify to NOT(P) since NOT(NOT$(P)) \equiv P$. Then, we can show the rule is not sound by giving a counterexample: $P =$ **false**. The antecedent is **true**, but the consequent is **false**.

(b) $$\frac{P \implies Q}{\text{NOT}(P) \text{ OR } Q}$$

Comments: Sound. We can show the inference rule is sound by filling out a truth table, and showing that the consequent is **true** for all rows where the antecedent is **true**:

Variables		Antecedent		Consequent
P	Q	$P \implies Q$	NOT(P)	OR(NOT$(P), Q)$
true	**true**	**true**	**false**	**true**
true	**false**	**false**	-	-
false	**true**	**true**	**true**	**true**
false	**false**	**true**	**true**	**true**

We use the - in entries that do not matter (since the antecedent is **false**), and all of the other rows the value of the consequent is **true**, so the inference rule is sound.

Problem 2 *Spot the Proof Bugs*

(a) Bogus Proof that 2102 = 2120.

1. We prove the proposition $P ::= 2102 = 2120$ using direct proof.
2. By the definition of even, 2102, which is even, can be written as $2k$ for some integer k.
3. By the definition of even, 2120, which is even, can be written as $2k$ for some integer k.
4. We know $2k = 2k$, and substituting from (2) on the left side and (3) on the right side gives, $2102 = 2120$.
5. Thus, we have proven P. □

Explain the flaw in the proof above: Comments: The problem is that we are using the definition of even twice, but with the same variable name. The definition of even is:

> An integer, z, is even if and only if there exists an integer k such that $z = 2k$.

The variable k in this definition represents some integer, but we cannot assume it is always the *same* integer, so need to use a different variable name if we use the definition twice for different integers.

(b) Bogus proof that $\sqrt{5}$ is irrational:

P: $\sqrt{5}$ is irrational.

1. We use proof by contradiction.
2. Suppose P is false: that the $\sqrt{5}$ is rational.
3. By the definition of rational, there exist two integers a and b such that $\sqrt{5} = \frac{a}{b}$
4. We can find an equal ratio that is in lowest terms, $\frac{c}{d} = \frac{a}{b}$
5. Using arithmetic,
   $\sqrt{5} = \frac{c}{d}$
   $5 = \frac{c^2}{d^2}$
   $5d^2 = c^2$
   So 5 divides c^2.
6. Using the Even Square Theorem, since 5 divides c^2, 5 also divides c.
7. ...

Explain the flaw in the proof above:

Comments: Step 6 of the prove uses the Even Square Theorem:

> **Even Square Theorem:** If n^2 is even, then n must be even.

The premise of the theorem is that n^2 is even, but here we are using it for $c^2 = 5d^2$ so the premise is not satisfied since c^2 is not known to be even.

Problem 3 *Product of Rationals*

Prove that the product of any two rational numbers must be a rational number.

Comments: We prove the proposition using a direct proof:

1. We prove the proposition $P ::=$ for any two rational numbers, x and y, the product xy is a rational number.
2. By the definition of rational number, we can write x as the ratio of two integers: $x = \frac{a}{b}$ where a and b are integers and b is non-zero.
3. By the definition of rational number, we can write y as the ratio of two integers $y = \frac{c}{d}$ where c and d are integers and d is non-zero.
4. By substitution and the rules of rational multiplication, $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
5. Since the product of two integers is an integer[1], we know ac is equal to some integer g.
6. Since the product of two non-zero integers is a non-zero integer, we know bd is equal to some non-zero integer h.
7. Thus, by substitution we can write the product xy as $\frac{g}{h}$ where g is an integer and h is a non-zero integer.
8. By the definition of a rational number, since $xy = \frac{g}{h}$ where g is an integer and h is a non-zero integer, we have shown that the product of any two rational numbers is a rational number.

[1] We use this without proof, but bonus points if you were able to prove this, which would require formally defining integer and integer multiplication, so is not something we expected.

Problem 4 *Proof of Irrationality*

Prove that the product of a positive rational number and an irrational number must be irrational. **Comments:** We prove the proposition using a proof by contradiction:

1. We prove the proposition $P ::=$ the product of a positive rational number, r, and an irrational number, x, rx must be an irrational number.
2. We prove by contradiction.
3. Suppose P is false. Then, there exists a positive rational number r and an irrational number x where their product xr is rational.
4. By the definition of rational, there exist two integers a, b such that $xr = \frac{a}{b}$ where b is non-zero.
5. By the definition of rational, there exist two integers c, d, such that $r = \frac{c}{d}$ where d is non-zero.
6. Since r is positive, we can assume that c is positive - if c and d are negative we can rewrite them as a $\frac{-c}{-d}$.
7. By substitution, $x \cdot \frac{c}{d} = \frac{a}{b}$.
8. By multiplication rules and algebra, $x = \frac{ad}{bc}$.
9. Since a, b, c, d are integers and we know the product of two integers is an integer[2], we know ad and bc are integers.
10. b is non-zero and c is positive, so bc must be non-zero.
11. We have expressed x as a ratio of two integers where the denominator is non-zero, so it is rational by definition.
12. This is a contradiction since in step 3 we supposed x was irrational. Therefore, P is true and the product of an irrational number and a positive rational number must be an irrational number. ■

[2]We use this without proof. Proving it would require careful definitions of the integers and multiplication, and would definitely be worth bonus points!

