Test 2 — Comments

Problem 1 *Basic Set Questions*

Write the value of each of the following expressions as an explicit set (showing all of the elements in the set if it is finite and using "..." if necessary and clear). You do not need to provide anything other than a correct description of the value of the expression for full credit on these problems.

(a) $\{0, 1, 2\} \cup \{\}$

Answer: $\{0, 1, 2\}$

(b) $\{2, 5\} \times \{10\}$

Answer: $\{(2, 10), (5, 10)\}$

(c) $\{(x, x + 1) \mid x \in \mathbb{N}, x \leq 4\}$

Answer: $\{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$

(d) $pow(\{9\})$

Answer: $\{\{\}, \{9\}\}$

(e) $\{\ |A| \mid A \in pow(\{0, 1, 2\})\ \}$

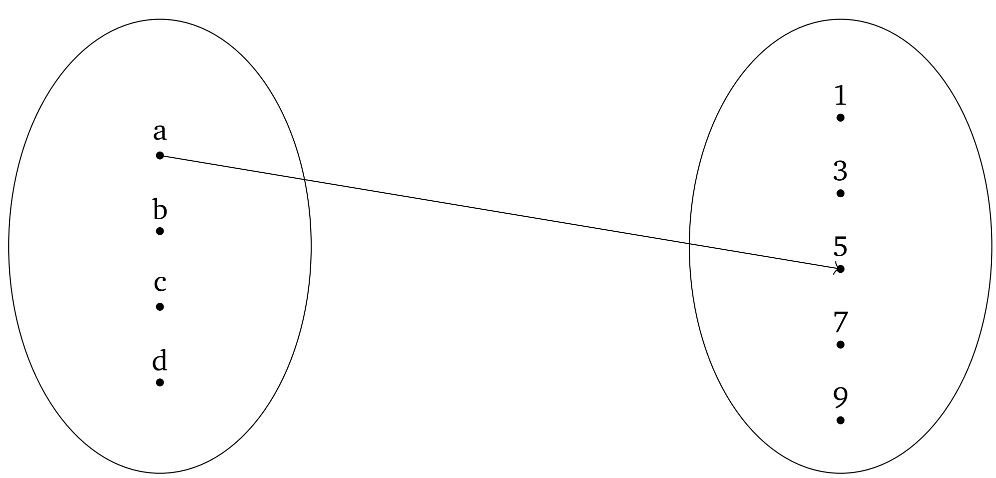
Comments: $\{0, 1, 2, 3\}$

Note that including duplicates is mathematically correct since $\{0, 1, 1, 1, 2, 2, 2, 3, 3, 3\}$ is the same set as $\{0, 1, 2, 3\}$, but we did not award full credit for answers with unnecessary duplicate elements since for full credit an answer should be clear and concise (not just mathematically correct).

Problem 2 *Binary Relation Graphs*

The diagram below depicts the relation, R:

$$R = (\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}, \{1, 3, 5, 7, 9\}, G = \{(a, 5)\})$$



(a) What is the *domain* of R?

Answer: $\{a, b, c, d\}$

(b) Either explain why it is not possible to make the relation *total* by adding or removing edges from the graph, or give a set of edges to add or remove that would make the relation *total*.

Answer: There are many possible answers, for example: Add $(b, 1)$, $(c, 1)$, $(d, 1)$.

(c) Either explain why it is not possible to make the relation *bijective* by adding or removing edges from the graph, or give a set of edges to add or remove that would make the relation *bijective*.

Answer: It is not possible to make the relation bijective, because the domain and codomain do not have the same cardinality.

Problem 3 *Binary Relation Properties*

Consider the following binary relation: $R = (\mathbb{N}, W, G = \{(x, 3x) \mid x \in \mathbb{N}\})$.

Give a value for W, the codomain, that makes R a bijection.

Answer: The definition of W should include just the elements of $\mathbb{N}$ that have in edges in G:

$$W ::= \{3x \mid x \in \mathbb{N}\}$$

Other acceptable ways of expressing the same set include:

$$W ::= \{x \in \mathbb{N} \mid 3 \,|\, x\}$$

or

$$W ::= \{0, 3, 6, 9, ...\}$$

Problem 4 *Set Proof*

For any sets A, B, and C prove $A \cup B \subseteq (A \cap C) \cup (C \cup B) \cup (A - C)$.

Comments: We want to prove $x \in A \cup B \implies x \in (A \cap C) \cup (C \cup B) \cup (A - C)$.

1. By the definition of union, $x \in A \cup B \iff x \in A \vee x \in B$, so we can prove the proposition by considering two cases, $x \in A$ and $x \in B$.
2. Case 1: $x \in A$. We split this into two sub-cases, based on whether or not $x \in C$:

   Case 1.1: $x \in A \wedge x \in C$**:** Then $x \in A \cap C$, by the definition of set intersection.
   Case 1.2: $x \in A \wedge x \notin C$**:** Then $x \in A - C$, by the definition of set difference.

3. Case 2: $x \in B$. By the definition of set union, every element of B is in $C \cup B$, so $x \in C \cup B$. The rest of the implicand just unions this set with other sets, so $x \in (A \cap C) \cup (C \cup B) \cup (A - C)$. ■

Problem 5 *Binary Relations Proof*

Prove that for any three finite sets, A, B, and C, if there exists a injective total function between A and B, and a injective total function between B and C, there must exist a injective total function between A and C.

Comments:

1. We prove the proposition that for any finite sets A, B, C, if there exists a injective total function, $R_{AB} = (A, B, G_{AB} \subseteq A \times B)$, and a injective total function $R_{BC} = (B, C, G_{BC} \subseteq B \times C)$, then there must exist a injective total function $R_{AC} = (A, C, G_{AC} \subseteq A \times C)$.
2. Our proof is to show how to construct R_{AC} such that it is an injective, total, function.
3. We define $G_{AC} = \{(a, c) \mid (a, b) \in G_{AB} \wedge (b, c) \in G_{BC}, a \in A, b \in B, c \in C\}$. Essentially, if there is an edge $(a, b) \in G_{AB}$ and an edge $(b, c) \in G_{BC}$ we will connect them together and form a new edge from a to c.
4. The binary relation R_{AC} is a injective, total function:

   R_{AC} **is *injective* (≤ 1 in):** Since R_{BC} is injective for any point $c \in C$, there must exist at most one in arrow $(b, c) \in B \times C$, and since R_{AB} is injective, there must be at most one in arrow $(a, b) \in A \times B$. If there does not exist any $b \in B$ such that $\exists a \in A.(a, b) \in G_{AB}$ and $(b, c) \in G_{BC}$, then our construction will not include any edge (a, c) so there are 0 arrows into c, which satisfies the injective property for c. If both of these arrows exist, our construction will create 1 arrow into c. Therefore from our construction, for any point $c \in C$ there exists at most one arrow $(a, c) \in A \times C$ so G_{AC} is injective.

   R_{AC} **is a *total function*:** Since G_{AB} is a total function, from every point a there is exactly one out arrow $(a, b) \in A \times B$, and since G_{BC} is a total function, from every point b there is exactly one out arrow $(b, c) \in B \times C$. These two arrows will form exactly 1 arrow in our construction. Therefore from our construction for any point $a \in A$ there will be exactly one arrow $(a, c) \in A \times C$ so G_{AC} is a total function.
5. We have shown R_{AC} is an injectiv total function, thus there exists a injective total function between A and C.

Problem 6 *Natural Numbers*

Consider the following alternative attempt to define the natural numbers:

A set Z is an *indonktive* set if and only if:

1. $\emptyset \in Z$.
2. If $x \in Z$ then *sucksessor*$(x) \in Z$.

Where for any set S, *sucksessor*$(S) ::= S \cup S$.

The set of the *natural numbers* is defined as:

$$\mathbb{M} = \{x \mid \text{for any indonktive set } Z, x \in Z\}$$

(Note that the only change from the definition of the natural numbers from class is how *sucksessor* is defined. The original definition was: *successor*$(S) ::= S \cup \{S\}$.)

Explain why this does **NOT** provide a definition that could be used to represent our familiar notion of the natural numbers:

Comments:

This definition of successor does **not** produce new mathematical object:

$$\begin{aligned}
&\text{“0”} := \emptyset \\
&\text{“1”} := \text{suckcessor(“0”)} = \text{suckcessor}(\emptyset) = \emptyset \cup \emptyset = \emptyset \\
&\text{“2”} := \text{suckcessor(“1”)} = \text{suckcessor(“0}'') = \emptyset \cup \emptyset = \emptyset \\
&\vdots
\end{aligned}$$

The union of an empty set with itself it the empty set, so the “sucksessor” function will always create more empty sets.

This is not a valid definition of the natural numbers because the set M only contains one mathematical object, $\emptyset$, so there is no way to represent the infinite set of natural numbers. Note that there are many *indonktive* sets, including the set of all the natural numbers (as defined using the correct definition), but the problem is that M is the set of elements that are in *all* indonktive sets. The only element that is in all indonktive sets is $\emptyset$, so M only has one element and cannot represent the natural numbers.

Optional Feedback

This question is **optional** and will not negatively impact your grade.

Do you feel your performance on this test will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the test, the course so far, or your hopes for the rest of the course here.)

Comments: 167 submissions did not provide any optional feedback.

35 submissions provided comments that indicated that they thought the exam was fair.

24 submissions wrote that they didn’t have as much time as they thought they needed to demonstrate how much they learned on the exam (presumably some of the 167 blank feedback submissions were also short on time). We do want to avoid time pressure in these tests, but also want to have enough questions to provide coverage of a variety of topics, include questions a enough different levels of difficult to measure gradations in understanding, and have at least a few questions that do require some evidence of deeper understanding needed to solve problems that require putting some ideas together. With a 50 minute class time for these exams, that is a lot to do in a limited time.

Of the remaining 14 submissions, there were two drawings (which we do appreciate, but are not qualified to judge if the smiley face or brain fart drawn indicate a high level of artistic talent), some comments about it being hard to know if the exam was fair until it is graded (which is a fair comments, but doesn’t provide much insight for us), and other helpful comments.

We didn't provide direct responses to your comments here, but did read them and do appreciate the feedback. Please do feel free to come to one of the instructor's office hours or make an appointment if there is anything you would like to discuss.

