Test 2

Name:	
UVA Computing ID:	
Enrolled Section (Fill in the correct circle):	◯ 10AM (David Evans) ◯ 2PM (Aidan San)
Honor Pledge (Sign your name):	On my honor as a student, I have neither given nor received unauthorized aid on this exam.

Instructions

– This exam is to be taken individually. You will have 50 minutes for this test.

– You are permitted to use a single paper page of notes (no larger than 8.5 x 11 inches) you prepared. You may not use any special devices (e.g., magnifying glasses) to read your page.

– No other resources, other than your own brain, body, writing instrument, and single note sheet are permitted during the exam.

– Write your UVA Computing ID (e.g., mst2k) clearly on the top of each page.

– Print clearly. We can only give credit for what we recognize as correct.

– Write your answers in the provided boxes. You are free to use the rest of each page for scratch paper. If you need more space for your answer (which you shouldn't), make sure it is clearly marked from the answer box where to find your answer.

– Please check all sides of the all the papers before you submit your exam.

Problem 1 *Basic Set Questions*

Write the value of each of the following expressions as an explicit set (showing all of the elements in the set if it is finite and using "..." if necessary and clear). You do not need to provide anything other than a correct description of the value of the expression for full credit on these problems.

(a) $\{0, 1, 2\} \cup \{\}$

(b) $\{2, 5\} \times \{10\}$

(c) $\{(x, x+1) \mid x \in \mathbb{N}, x \leq 4\}$

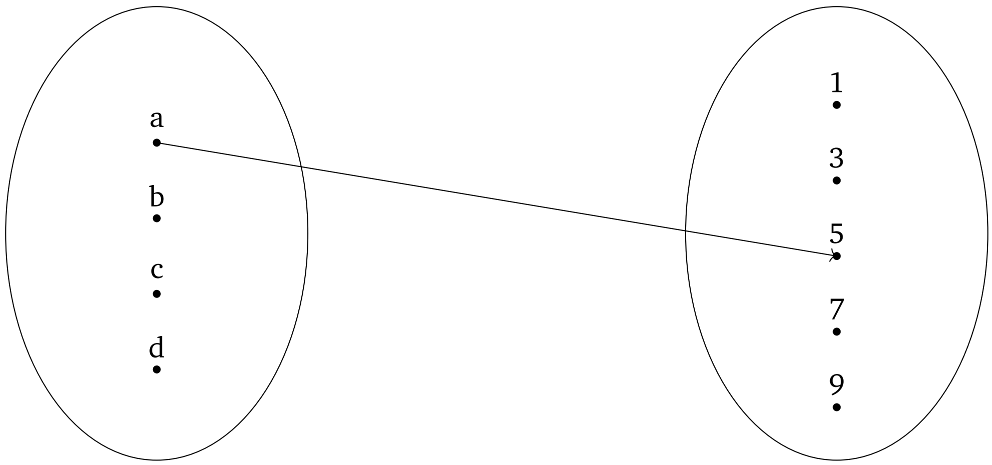
(d) $pow(\{9\})$

(e) $\{\ |A| \mid A \in pow(\{0, 1, 2\})\ \}$

Problem 2 *Binary Relation Graphs*

The diagram below depicts the relation, R:

$$R = (\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}, \{1, 3, 5, 7, 9\}, G = \{(a, 5)\})$$



(a) What is the *domain* of R?

(b) Either explain why it is not possible to make the relation *total* by adding or removing edges from the graph, or give a set of edges to add or remove that would make the relation *total*.

(c) Either explain why it is not possible to make the relation *bijective* by adding or removing edges from the graph, or give a set of edges to add or remove that would make the relation *bijective*.

Problem 3 *Binary Relation Properties*

Consider the following binary relation: $R = (\mathbb{N}, W, G = \{(x, 3x) \mid x \in \mathbb{N}\})$.

Give a value for W, the codomain, that makes R a bijection.

Problem 4 *Set Proof*

For any sets A, B, and C prove $A \cup B \subseteq (A \cap C) \cup (C \cup B) \cup (A - C)$.

Problem 5 *Binary Relations Proof*

Prove that for any three finite sets, A, B, and C, if there exists a injective total function between A and B, and a injective total function between B and C, there must exist a injective total function between A and C.

Problem 6 *Natural Numbers*

Consider the following alternative attempt to define the natural numbers:

> A set Z is an *indonktive* set if and only if:
>
> 1. $\emptyset \in Z$.
> 2. If $x \in Z$ then *sucksessor*$(x) \in Z$.
>
> Where for any set S, *sucksessor*$(S) ::= S \cup S$.

The set of the *natural numbers* is defined as:

$$\mathbb{M} = \{x \mid \text{for any indonktive set } Z, x \in Z\}$$

(Note that the only change from the definition of the natural numbers from class is how *sucksessor* is defined. The original definition was: *successor*$(S) ::= S \cup \{S\}$.)

Explain why this does **NOT** provide a definition that could be used to represent our familiar notion of the natural numbers:

End of Test 2! The last page is optional.
Please check that you filled in your UVA Computing ID on each page.

Optional Feedback

This question is **optional** and will not negatively impact your grade.

Do you feel your performance on this test will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the test, the course so far, or your hopes for the rest of the course here.)

Definitions and Theorems

This page provides definitions we use in the exam problems and theorems that you are free to use without needing to restate or prove.

Binary Relations

Binary Relation: A *binary relation*, R, consists of a set, A, called the *domain* of R, a set, B, called the *codomain* of R, and a subset of $A \times B$ called the *graph*, G, of R: $R = (A, B, G \subseteq A \times B)$.

function: A binary relation is a *function* iff it has the [≤ 1 arrow **out**] property which means

$$\forall a \in A \,.\, |\{(a, b) \mid b \in B, (a, b) \in G\}| \leq 1$$

surjective: A binary relation is *surjective* iff it has the [≥ 1 arrows **in**] property.

total: A binary relation is *total* iff it has the [≥ 1 arrows **out**] property.

injective: A binary relation is *injective* iff it has the [≤ 1 arrow **in**] property.

bijective: A binary relation is *bijective* iff it has both the [$= 1$ arrow **out**] and the [$= 1$ arrow **in**] property.

Sets

Empty Set: $\emptyset$ or $\{\}$ are equivalent and both refer to an empty set.

Membership: for any set S, $x \in S \iff x$ is in the set S.

Non-membership: for any set S, $x \notin S \iff \neg(x \in S)$.

Subset: for any sets A and B, $A \subseteq B \iff \forall x \in A, x \in B$.

Equality: for any sets A and B, $A = B \iff A \subseteq B \land B \subseteq A$.

Complement: for any set S, $x \in \overline{S} \iff (x \in D \land x \notin S)$ where D is the domain of discourse.

Union: for any sets A and B, $x \in A \cup B \iff x \in A \lor x \in B$.

Intersection: for any sets A and B, $x \in A \cap B \iff x \in A \land x \in B$.

Difference: for any sets A and B, $x \in A - B \iff x \in A \land x \notin B$.

Powerset: for any sets A and B, $B \in pow(A) \iff B \subseteq A$.

Cardinality: The *cardinality* of the set $\mathbb{N}_k = \{n \in \mathbb{N} \mid n < k\}$ is k. Two sets have the *same cardinality* if and only if there is a bijection between the two sets.

Injective Size Theorem: For any sets A and B, if there is a total injective function between A and B then $|A| \leq |B|$.

for all: For any set S, $\forall x \in S \,.\, P(x)$ is a proposition that is true if the proposition P is true for every element of the set S.

exists: For any set S, $\exists x \in S \,.\, P(x)$ is a proposition that is true if the proposition P is true for at least one element of the set S.

Natural Numbers

Inductive: A set Z is an *inductive* set if and only if (1) $\emptyset \in Z$, and (2) If $x \in Z$ then *successor*$(x) \in Z$ where for any set S, *successor*$(S) ::= S \cup \{S\}$.

Natural Numbers: The set of the *natural numbers* is defined as $\mathbb{N} = \{x \mid \text{for any inductive set } Z, x \in Z\}$.