Test 3 — Comments

Problem 1 *Well and Unwell Ordered Sets*

For each subproblem below, select the best answer to the question

> "Is the given set well ordered by the comparator, with the standard equality operator $=$?"

and support your answer with a brief, but clear and convincing, argument.

(a) Set: $\mathbb{Z}$ (the integers); Comparator: $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ✓ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

Since the integers continue forever in both directions, there is no minimum element if $\mathbb{Z}$ using the $<$ comparator since we can always find a lower integer.

(b) Set: $\{n \in \mathbb{N} \mid n \text{ is divisible by } 7\}$; Comparator: $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ✓ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

We know this set is well-ordered by $<$ since it is a subset of $\mathbb{N}$ and we know $\mathbb{N}$ is well-ordered by $<$.

(c) Set: $\emptyset$; Comparator: $\geq$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ✓ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

Well-ordering is always (vacuously) true on an empty set, since it is a property of all non-empty subsets of the set, and $\emptyset$ has no non-empty subsets.

(d) Set: $\{(x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}\}$; Comparator: $smaller((x_1, y_1), (x_2, y_2)) ::= x_1 < x_2$ or $y_1 < y_2$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ✓ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

The definition of $smaller$ does not satisfy the trichotomy property, which is required for an ordering. For example, $(1, 2) < (2, 1)$ and $(2, 1) < (1, 2)$.

(e) Set: $\{\sum_{i=0}^{k} i \mid k \in \mathbb{N}\}$; $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ✓ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

We know the set is well-ordered by $<$ since it is a subset of $\mathbb{N}$. Since $\mathbb{N}$ is well-ordered all of its subsets have a minimum element, and all of the subsets of any subset of $\mathbb{N}$ are also subsets of $\mathbb{N}$. Thus, we know any subset of $\mathbb{N}$ is well-ordered by $<$.

Problem 2 *Variants of Induction Principles*

For each of the possible induction proof variants, answer whether or not the proof method is valid. A proof method is valid if all proofs using the method correctly prove the stated proposition.

(a)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m)$.

✓ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

In step 2, $P(m)$ is used to prove itself, so $P(1)$ and higher are never proven.

(b)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, \neg P(m+1) \implies \neg P(m)$.

◯ Not a valid proof method

✓ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

This simply takes induction and uses the contrapositive inference rule which is sound so it is valid. The by the contrapositive rule (or just by checking the truth table), the implication $\neg P(m+1) \implies \neg P(m)$ implies $P(m) \implies P(m+1)$, our standard inductive step for induction on the natural numbers.

(c)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m+2)$.

✓ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

$P(1)$ is never proven (it is just skipped going from 0 to 2). This proof method would prove $n \in Evens.P(n)$, but does not prove anything about the odd natural numbers, so is not valid for proving $\forall n \in \mathbb{N}.P(n)$.

(d)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies (P(2m) \land P(2m+1))$.

◯ Not a valid proof method

✓ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

Yes, it is strange but valid! The inductive step does cover all the natural numbers, since all numbers are either even or odd. To see this, build up from the base value of 0 (proven by step one). Then we get $P(0) \implies (P(0) \land P(1))$ which proves $P(1)$. Next $P(1) \implies (P(2) \land P(3))$, $P(2) \implies (P(4) \land P(5))$, and so on to cover all the natural numbers and prove $\forall n \in \mathbb{N} \,.\, P(n)$.

$\star$ We will give a challenge bonus to the first student to show an actual valid and sensible (at least not totally contrived) proof of a non-trivial theorem using this method.

Problem 3 *Betable Numbers*

The Primes-Under-Five Casino has only two different chips, \$2 and \$3, but an unlimited supply of each. We have started a proof that every natural number value bet above \$1 can be made using a combination of \$2 and \$3 chips. Complete the proof by filling in the missing parts.

1. We prove the proposition, $\forall n \in \mathbb{N} \geq 2, P(n)$ where:

$$P(n) := \exists a, b \in \mathbb{N} \,.\, n = 2a + 3b.$$

   using the well ordering principle.

2. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} \mid n \geq 2 \text{ and } \neg P(n)\}$.
3. Assume (towards a contradiction) that C is non-empty.
4. By the well ordering principle, there exists an $m \in C$ that is the minimum of C.
5. We know the minimum counter example is ≥ 4, because $P(2)$ and $P(3)$ are both True.

   $P(2)$**:** We can make the \$2 bet with one \$2 chip: $2 = 2a + 3b$ where $a = 1$ and $b = 0$.
   $P(3)$**:** We can make the \$3 bet with one \$3 chip: $3 = 2a + 3b$ where $a = 0$ and $b = 1$.

6. So we know $m \geq 4$, and we know $P(m-2)$ since otherwise m would not be the minimum in C, so $\exists a, b \,.\, m - 2 = 2a + 3b$.
7. We can add one \$2 chip: $m - 2 + 2 = 2a + 3b + 2$, and by algebra, $m = 2(a+1) + 3b$.
8. Since $a + 1 \in \mathbb{N}$, this shows $P(m)$, contradicting $m \in C$.
9. Thus, C must be empty, and $P(n)$ holds for all $n \geq 2$. ■

Problem 4 *Texas Takeaway Game*

In the proof, we are proving a slightly modified version of the take-away game. In the takeaway game from class, each player is allowed to take $r \in \{1, 2, 3\}$ sticks. In the popular Texas variant, each player can only take one or two sticks: $r \in \{1, 2\}$. Fill in the box to complete the proof below that Texas takeaway always finishes:

1. We are proving $\forall n \in \mathbb{N}$, a Texas takeaway game with n sticks finishes.
2. We prove using the principle of strong induction on the natural numbers where $P(n) ::=$ the Texas takeaway game $(n,\, p \in \{\text{true}, \text{false}\})$ finishes.
3. First, we prove **Base cases:**

   $P(0)$**:** a game with 0 sticks is finished by definition.

   $P(1)$**:** for a game with 1 stick, the player has one possible move, to take one stick. This leaves 0 sticks, and we know the game finishes because we already proved $P(0)$.

   $P(2)$**:** for a game with 2 sticks, the player has two possible moves: (1) take one stick, leaving one stick, which we know finished by $P(1)$; or (2) take two sticks, leaving none, which we know finishes by $P(0)$.
4. Next, the **inductive step:** $\forall m \geq 2 \in \mathbb{N}\ .\ \bigwedge_{k=0}^{m} P(k) \implies P(m+1)$.

4. We consider the two possible elements of *ValidMoves*$((m+1, p))$:

   Case 1: $(m, \neg p)$: Finishes because of $P(m)$.

   Case 2: $(m-1, \neg p)$: Finishes because of $P(m-1)$.

5. This proves $P(m+1)$.
6. By the **principle of strong induction on the natural numbers**, we have proven $\forall n \in \mathbb{N}\ .\ P(n)$, so we know a Texas takeaway game with n sticks always finishes.

Problem 5 *Sum of Powers*

Prove the following theorem:

For any natural number n,

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

(Note that an empty summation is defined as 0, so $\sum_{i=0}^{-1} 1 = 0$.)

You can use as much space as you want for this, continuing on to the next page.

Proof using Well Ordering Principle:

1. We are proving the proposition, $P(n) ::= \sum_{i=0}^{n-1} 2^i = 2^n - 1$.
2. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} \mid \neg P(n)\}$.
3. Assume towards a contradiction that C is non-empty.
4. By the well ordering principle, there exists an $m \in C$ that is the minimum of C.
5. We know $m \geq 1$, because $\sum_{i=0}^{-1} 2^i = 0$ (an empty sum) and $2^0 - 1 = 1 - 1 = 0$.
6. Since $m \geq 1$, we know $m - 1 \in \mathbb{N}$ and $m - 1 \geq 0$.
7. Since $m - 1 < m$, and m was the minimum of C, we know $m - 1 \notin C$, so $P(m-1)$ must be true meaning:
$$\sum_{i=0}^{m-2} 2^i = 2^{m-1} - 1$$
8. Then, adding 2^{m-1} to both sides we get:
$$\begin{aligned}\sum_{i=0}^{m-2} 2^i + 2^{m-1} &= 2^{m-1} - 1 + 2^{m-1} \\ \sum_{i=0}^{m-1} 2^i &= 2^{m-1} - 1 + 2^{m-1} \\ \sum_{i=0}^{m-1} 2^i &= 2 \cdot 2^{m-1} - 1 \\ \sum_{i=0}^{m-1} 2^i &= 2^m - 1\end{aligned}$$
9. But this means that $P(m)$ is true, which contradicts $m \in C$.
10. Thus, C must be empty, and $P(n)$ holds for all $n \in \mathbb{N}$. ■

Proof using the Principle of Induction:

1. We prove using the principle of induction on the natural numbers.
2. Our inductive hypothesis (predicate), $P(n) ::= \sum_{i=0}^{n-1} 2^i = 2^n - 1$
3. **Base case**: $P(0)$
   $\sum_{i=0}^{-1} 2^i = 0$ (an empty sum) and $2^0 - 1 = 1 - 1 = 0$.
4. **Inductive Step**: $\forall m \in \mathbb{N} \, . \, P(m) \implies P(m+1)$. We prove the non-vacuous part of the implication, where $P(m)$ is true.
5. We assume $P(m)$, so
$$\sum_{i=0}^{m-1} 2^i = 2^m - 1$$
6. We take $P(m)$ and add 2^m to both sides:
$$\begin{aligned} \sum_{i=0}^{m-1} 2^i + 2^m &= 2^m - 1 + 2^m \\ \sum_{i=0}^{m} 2^i &= 2^m - 1 + 2^m \\ \sum_{i=0}^{m} 2^i &= 2 \cdot 2^m - 1 \\ \sum_{i=0}^{m} 2^i &= 2^{m+1} - 1 \end{aligned}$$
7. It follows by induction that $P(n)$ is true for all natural numbers.

End of Test 3 — Comments! The last page is optional.
Please check that you filled in your UVA Computing ID on each page.

Optional Feedback

This question is **optional** and will not negatively impact your grade.

Do you feel your performance on this test will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the test, the course so far, or your hopes for the rest of the course here, or to draw a picture depicting your favorite indonktive creature.)

Definitions and Theorems

This page provides definitions we use in the exam problems and theorems that you are free to use without needing to restate or prove. You may (carefully) rip off this page and use it during the exam.

Definition of ordered: A set S is *ordered* with respect to some comparator, $<: S \times S \to Boolean$, and equality operator, $=: S \times S \to Boolean$, iff $\forall a, b, c \in S$:

1. $\neg(a = b) \implies (a < b) \vee (b < a)$.
2. $(a < b) \wedge (b < c) \implies a < c$.

Definition of well ordered: An ordered set S with comparator $<$ and equality operator $=$, is *well ordered* iff all of its non-empty subsets have a minimum element:

$$\forall T \subseteq S \,.\, T \neq \emptyset \implies \exists m \in T \,.\, \forall t \in T \,.\, t \neq m \implies m < t.$$

Well Ordering Principle: Every *non-empty* set of natural numbers has a minimum.

Principle of Induction on the Natural Numbers: To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m+1)$.

Principle of Strong Induction on the Natural Numbers: To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, \bigwedge_{k=0}^{m} P(k) \implies P(m+1)$.

Summation Notation: The notation

$$\sum_{i=0}^{m} f(i)$$

means

$$f(0) + f(1) + \cdots + f(m).$$

Big And Notation: The notation

$$\bigwedge_{i=0}^{m} P(i)$$

means

$$P(0) \wedge P(1) \wedge \cdots \wedge P(m).$$