Test 3

Name:	
UVA Computing ID:	
Enrolled Section (Fill in the correct circle):	◯ 10AM (David Evans) ◯ 2PM (Aidan San)
Honor Pledge (Sign your name):	On my honor as a student, I have neither given nor received unauthorized aid on this exam.

Instructions

– This exam is to be taken individually. You will have 50 minutes for this test.

– You are permitted to use a single paper page of notes (no larger than 8.5 x 11 inches) you prepared. You may not use any special devices (e.g., magnifying glasses) to read your page.

– No other resources, other than your own brain, body, writing instrument, and single note sheet are permitted during the exam.

– Write your UVA Computing ID (e.g., mst2k) clearly on the top of each page.

– Print clearly. We can only give credit for what we recognize as correct.

– Write your answers in the provided boxes. You are free to use the rest of each page for scratch paper. If you need more space for your answer (which you shouldn't), make sure it is clearly marked from the answer box where to find your answer.

– Please check all sides of the all the papers before you submit your exam.

Problem 1 *Well and Unwell Ordered Sets*

For each subproblem below, select the best answer to the question

> "Is the given set well ordered by the comparator, with the standard equality operator $=$?"

and support your answer with a brief, but clear and convincing, argument.

(a) Set: $\mathbb{Z}$ (the integers); Comparator: $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

(b) Set: $\{n \in \mathbb{N} \mid n \text{ is divisible by } 7\}$; Comparator: $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

(c) Set: $\emptyset$; Comparator: $\geq$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

(d) Set: $\{(x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}\}$; Comparator: $smaller((x_1, y_1), (x_2, y_2)) ::= x_1 < x_2$ or $y_1 < y_2$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

(e) Set: $\{\sum_{i=0}^{k} i \mid k \in \mathbb{N}\}$; $<$.

- ◯ Not a Valid Question (either the set or the comparator is not reasonable)
- ◯ Not Ordered
- ◯ Ordered, but Not Well Ordered
- ◯ Well Ordered

Justify your answer with a brief but clear and convincing explanation:

Problem 2 *Variants of Induction Principles*

For each of the possible induction proof variants, answer whether or not the proof method is valid. A proof method is valid if all proofs using the method correctly prove the stated proposition.

(a)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m)$.

◯ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

(b)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, \neg P(m+1) \implies \neg P(m)$.

◯ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

(c)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m+2)$.

◯ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

(d)

To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies (P(2m) \land P(2m+1))$.

◯ Not a valid proof method

◯ Valid proof method

Justify your answer with a brief but clear and convincing explanation:

Problem 3 *Betable Numbers*

The Primes-Under-Five Casino has only two different chips, \$2 and \$3, but an unlimited supply of each. We have started a proof that every natural number value bet above \$1 can be made using a combination of \$2 and \$3 chips. Complete the proof by filling in the missing parts.

1. We prove the proposition, $\forall n \in \mathbb{N} \geq 2, P(n)$ where:

$$P(n) := \exists a, b \in \mathbb{N} \,.\, n = 2a + 3b.$$

   using the well ordering principle.

2. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} \mid n \geq 2 \text{ and } \neg P(n)\}$.
3. Assume (towards a contradiction) that C is non-empty.
4. By the well ordering principle, there exists an $m \in C$ that is the minimum of C.

5.

6. So we know $m \geq 4$, and we know $P(m-2)$ since otherwise m would not be the minimum in C, so $\exists a, b \,.\, m - 2 = 2a + 3b$.
7. We can add one \$2 chip: $m - 2 + 2 = 2a + 3b + 2$, and by algebra, $m = 2(a+1) + 3b$.
8. Since $a + 1 \in \mathbb{N}$, this shows $P(m)$, contradicting $m \in C$.
9. Thus, C must be empty, and $P(n)$ holds for all $n \geq 2$. ∎

Problem 4 *Texas Takeaway Game*

In the proof, we are proving a slightly modified version of the take-away game. In the takeaway game from class, each player is allowed to take $r \in \{1, 2, 3\}$ sticks. In the popular Texas variant, each player can only take one or two sticks: $r \in \{1, 2\}$. Fill in the box to complete the proof below that Texas takeaway always finishes:

1. We are proving $\forall n \in \mathbb{N}$, a Texas takeaway game with n sticks finishes.
2. We prove using the principle of strong induction on the natural numbers where $P(n) ::=$ the Texas takeaway game $(n,\, p \in \{\text{true}, \text{false}\})$ finishes.
3. First, we prove **Base cases:**

   $P(0)$**:** a game with 0 sticks is finished by definition.

   $P(1)$**:** for a game with 1 stick, the player has one possible move, to take one stick. This leaves 0 sticks, and we know the game finishes because we already proved $P(0)$.

   $P(2)$**:** for a game with 2 sticks, the player has two possible moves: (1) take one stick, leaving one stick, which we know finished by $P(1)$; or (2) take two sticks, leaving none, which we know finishes by $P(0)$.
4. Next, the **inductive step:** $\forall m \geq 2 \in \mathbb{N}\ .\ \bigwedge_{k=0}^{m} P(k) \implies P(m+1)$.

5. This proves $P(m+1)$.
6. By the **principle of strong induction on the natural numbers**, we have proven $\forall n \in \mathbb{N}\ .\ P(n)$, so we know a Texas takeaway game with n sticks always finishes.

Problem 5 *Sum of Powers*

Prove the following theorem:

For any natural number n,

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

(Note that an empty summation is defined as 0, so $\sum_{i=0}^{-1} 1 = 0$.)

You can use as much space as you want for this, continuing on to the next page.

End of Test 3! The last page is optional.
Please check that you filled in your UVA Computing ID on each page.

Optional Feedback

This question is **optional** and will not negatively impact your grade.

Do you feel your performance on this test will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the test, the course so far, or your hopes for the rest of the course here, or to draw a picture depicting your favorite indonktive creature.)

Definitions and Theorems

This page provides definitions we use in the exam problems and theorems that you are free to use without needing to restate or prove. You may (carefully) rip off this page and use it during the exam.

Definition of ordered: A set S is *ordered* with respect to some comparator, $<: S \times S \to Boolean$, and equality operator, $=: S \times S \to Boolean$, iff $\forall a, b, c \in S$:

1. $\neg(a = b) \implies (a < b) \vee (b < a)$.
2. $(a < b) \wedge (b < c) \implies a < c$.

Definition of well ordered: An ordered set S with comparator $<$ and equality operator $=$, is *well ordered* iff all of its non-empty subsets have a minimum element:

$$\forall T \subseteq S \,.\, T \neq \emptyset \implies \exists m \in T \,.\, \forall t \in T \,.\, t \neq m \implies m < t.$$

Well Ordering Principle: Every *non-empty* set of natural numbers has a minimum.

Principle of Induction on the Natural Numbers: To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m+1)$.

Principle of Strong Induction on the Natural Numbers: To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, \bigwedge_{k=0}^{m} P(k) \implies P(m+1)$.

Summation Notation: The notation

$$\sum_{i=0}^{m} f(i)$$

means

$$f(0) + f(1) + \cdots + f(m).$$

Big And Notation: The notation

$$\bigwedge_{i=0}^{m} P(i)$$

means

$$P(0) \wedge P(1) \wedge \cdots \wedge P(m).$$