Test 4

Name:	
UVA Computing ID:	
Enrolled Section (Fill in the correct circle):	◯ 10AM (David Evans) ◯ 2PM (Aidan San)
Honor Pledge (Sign your name):	On my honor as a student, I have neither given nor received unauthorized aid on this exam.

Instructions

– This exam is to be taken individually. You will have 50 minutes for this test.

– You are permitted to use a single paper page of notes (no larger than 8.5 x 11 inches) you prepared. You may not use any special devices (e.g., magnifying glasses) to read your page.

– No other resources, other than your own brain, body, writing instrument, and single note sheet are permitted during the exam.

– Write your UVA Computing ID (e.g., mst2k) clearly on the top of each page.

– Print clearly. We can only give credit for what we recognize as correct.

– Write your answers in the provided boxes. You are free to use the rest of each page for scratch paper. If you need more space for your answer (which you shouldn't), make sure it is clearly marked from the answer box where to find your answer.

– Please check all sides of the all the papers before you submit your exam.

Problem 1 *Cardinalities*

For each subproblem, answer if S is *finite*, *countably infinite* or *uncountable* and give a brief explanation why

(a) $S := \{\textsf{Aidan}, \textsf{Dave}\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(b) $S := \mathbb{R} - \mathbb{N}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(c) $S := (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

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(d) $S := \mathbb{R} - pow(\mathbb{N})$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(e) $S := \{G \mid G \in pow(\mathbb{N} \times \mathbb{N}) \wedge R = (\mathbb{N}, \mathbb{N}, G) \text{ is a bijection}\}$

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

(f) $S :=$ the set of all surjective functions between $\mathbb{N}$ and $\mathbb{N}$.

- ◯ S is finite
- ◯ S is countably infinite
- ◯ S is uncountable

Justify your answer with a brief but clear and convincing explanation:

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Problem 2 *Diagonalization Proof?*

In class we saw a diagonalization proof (very close to the one Cantor developed in his 1891 paper) that the set of infinite binary strings, $\{0,1\}^\infty$ is *uncountable* (this is identical to the proof in the practice problems)

1. We prove the set of infinite binary strings, $S = \{0,1\}^\infty$ is uncountable by contradiction.
2. Assume towards a contradiction that there exists a surjective function $R = (\mathbb{N}, S, G \subseteq \mathbb{N} \times S)$ from $\mathbb{N}$ to S.
3. Since R is function, for each $n \in \mathbb{N}$ there is at most one $s \in S$ where $(n, s) \in G$. Let $A_n := s$ be that element and identify the characters of A_n as $a_{n,1}a_{n,2}a_{n,3}\ldots$.
4. Define $b = b_1b_2b_3\ldots$ where $b_i = \neg a_{(i,i)}$ and $\neg 0 = 1$ and $\neg 1 = 0$.
5. Since b is an infinite bitstring, we know $b \in S$. Since each bit is different from one bit in each A_n and since R is surjective we know $\forall s \in S. \exists m \in \mathbb{N}. A_m = s$ but b is not in the mapping and we have a contradiction.

For each subproblem, indicate of the alternate construction for step 4 would result in a valid or invalid proof and explain why. (Differences from the previous subproblem are underlined.)

(a) Define $b = b_1b_2b_3\ldots$ where $b_i = W(a_{i,i})$ where $W(0) = 5$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

(b) Define $b = b_1b_2b_3\ldots$ where $b_i = W(a_{\underline{i+1,i+1}})$ where $W(0) = \underline{1}$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

(c) Define $b = b_1\underline{c_1}b_2\underline{c_2}b_3\underline{c_3}\ldots$ where $b_i = W(a_{\underline{i,i}})$ <u>and $c_i = W(a_{i,2i})$</u> where $W(0) = 1$ and $W(1) = 0$.

Justify your answer with a brief but clear and convincing explanation:

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Problem 3 *True, False, or Unresolvable*

For each subproblem, indicate the truthiness of the stated proposition and provide a brief but clear and convincing justification for your answer.

(a) There exists a surjective function from the natural numbers to the reals $R = (\mathbb{N}, \mathbb{R}, G)$.

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

(b) There exists a G for which $R = (\mathbb{N}, pow(\mathbb{N}), G \subseteq \mathbb{N} \times pow(\mathbb{N}))$ is total (≥ 1 out) and injective (≤ 1 in).

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

(c) $|pow(pow(\mathbb{R}))| > |pow(\mathbb{R})|$.

- ◯ True
- ◯ False
- ◯ Unresolvable in ZFC

Justify your answer with a brief but clear and convincing explanation:

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Problem 4 *Subsets of* $\mathbb{N}$

Define S_n as the set of all n-element subsets of $\mathbb{N}$:

$$S_n := \{s | s \in pow(\mathbb{N}) \land |s| = n\}$$

(a) $P := \forall n \in \mathbb{N} \,.\, S_n$ is ________ (select the *one* true option to complete the proposition)

- ◯ Finite
- ◯ Countable
- ◯ Countably Infinite
- ◯ Uncountable

(b) Prove the proposition P (as you completed it in part a). (We don't think you need more space than the rest of this page to complete your proof, but it's okay if you go onto the next page.)

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End of Test 4! The last page is optional.
Please check that you filled in your UVA Computing ID on each page.

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Optional Feedback

This question is **optional** and will not negatively impact your grade.

Do you feel your performance on this test will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the test, the course so far, or your hopes for the rest of the course here, or to draw a picture depicting your favorite uncountable vaguabest.)

Is there anything else you think we should know as the end of the semester approaches?

Definitions and Theorems

This page provides definitions we use in the exam problems and theorems that you are free to use without needing to restate or prove. You may (carefully) rip off this page and use it during the exam.

Definitions

Cardinality: The *cardinality* of the set $\mathbb{N}_k = \{n \in \mathbb{N} \mid n < k\}$ is k. Two sets have the *same cardinality* if and only if there is a bijection between the two sets.

Finite: A set S is *finite*, if and only if there is a **bijection** between S and some $\mathbb{N}_k$.

Countable: A set S is *countable*, if and only if there is a surjective function [≤ 1 out, ≥ 1 in] from $\mathbb{N}$ to S.

Countably infinite: A set S is *countably infinite* (infinite and countable) if and only if there is a bijection between S and $\mathbb{N}$.

Binary Relation: A *binary relation*, R, consists of a set, A, called the *domain* of R, a set, B, called the *codomain* of R, and a subset of $A \times B$ called the *graph*, G, of R: $R = (A, B, G \subseteq A \times B)$.

function: A binary relation is a *function* iff it has the [≤ 1 arrow **out**] property which means

$$\forall a \in A \,.\, |\{(a, b) \mid b \in B, (a, b) \in G\}| \leq 1$$

surjective: A binary relation is *surjective* iff it has the [≥ 1 arrows **in**] property.

total: A binary relation is *total* iff it has the [≥ 1 arrows **out**] property.

injective: A binary relation is *injective* iff it has the [≤ 1 arrow **in**] property.

bijective: A binary relation is *bijective* iff it has both the [$= 1$ arrow **out**] and the [$= 1$ arrow **in**] property.

Subset: for any sets A and B, $A \subseteq B \iff \forall x \in A, x \in B$.

Union: for any sets A and B, $x \in A \cup B \iff x \in A \lor x \in B$.

Difference: for any sets A and B, $x \in A - B \iff x \in A \land x \notin B$.

Cartesian Product: for any sets A and B, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Powerset: for any sets A and B, $B \in$ *pow*$(A) \iff B \subseteq A$.

Theorems

You may use the following theorems without proof:

Cantor's Theorem. For all sets A, $|pow(A)| > |A|$.

The following sets are *countably infinite*: $\mathbb{N}$ (natural numbers); $\mathbb{N} \times \mathbb{N}$; $\mathbb{Z}$ (integers); $\mathbb{Q}$ (rational numbers); $\{0, 1\}^*$ (finite binary strings).

The following sets are *uncountable*: *pow*$(\mathbb{N})$; $\mathbb{R}$ (real numbers); $\{0, 1\}^\infty$ (infinite binary strings); the set of permutations of $\mathbb{N}$; the set of all bijections between $\mathbb{N}$ and $\mathbb{N}$.

Principle of Induction on the Natural Numbers: To prove $\forall n \in \mathbb{N} \,.\, P(n)$:

1. Prove $P(0)$.
2. Prove $\forall m \in \mathbb{N} \,.\, P(m) \implies P(m + 1)$.